

Curves on Oeljeklaus-Toma Manifolds

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Abstract

Oeljeklaus-Toma manifolds are complex non-Kähler manifolds constructed by Oeljeklaus and Toma from certain number fields, and generalizing the Inoue surfaces S_m . We prove that Oeljeklaus-Toma manifolds contain no compact complex curves.

Contents

1 Introduction.	1
1.1 Oeljeklaus-Toma manifolds	1
2 Curves on the Oeljeklaus-Toma manifolds	2
2.1 The exact positive (1,1)-form on the Oeljeklaus-Toma manifold	2
2.2 (1,1)-form ω and curves on the Oeljeklaus-Toma manifold	3

1 Introduction.

Oeljeklaus-Toma manifolds (defined in [O–T]) are compact complex manifolds that are generalizing Inoue surfaces (defined in [I]). Let us describe them in detail.

1.1 Oeljeklaus-Toma manifolds

Let K be a number field (a finite extension of \mathbb{Q}), s be a number of its real embeddings and $2t$ be number of its complex embeddings. One can easily prove that for each s and t there exists a field K which has these numbers of real and complex embeddings (see e.g. [O–T]).

Definition 1.1: *Ring of algebraic integers* O_K is a subring of K that consists of all roots of polynomials with integer coefficients which lie in K . *Unit group* O_K^* is a multiplicative subgroup of invertible elements of O_K .

Let m be $s + t$. Let $\sigma_1, \dots, \sigma_s$ be real embeddings of the field K , $\sigma_{s+1}, \dots, \sigma_{s+2t}$ be complex embeddings such that σ_{s+i} and σ_{s+t+i} are complex conjugate for each i from 1 to t . Now we can define a map $l : O_K^* \rightarrow \mathbb{R}^m$ where $l(u) = (\ln |\sigma_1(u)|, \dots, \ln |\sigma_s(u)|, 2 \ln |\sigma_{s+1}(u)|, \dots, 2 \ln |\sigma_m(u)|)$.

Denote $O_k^{*,+} = \{a \in O_K^* : \sigma_i(a) > 0, i = 1, \dots, s\}$. Let's consider following definitions:

Definition 1.2: A *lattice* Λ in \mathbb{R}^n is a discrete additive subgroup such that $\Lambda \otimes \mathbb{R} = \mathbb{R}^n$.

Definition 1.3: [O–T] Group $U \subset O_k^{*,+}$ of rank s is called *admissible for the field K* if the projection of $l(U)$ to the first s components is a lattice in \mathbb{R}^s .

Consider a linear space $L = \{x \in \mathbb{R}^m \mid \sum_{i=1}^m x_i = 0\}$. The projection of $L \subset \mathbb{R}^m$ to the first s coordinates is surjective, because $s < m$. Using the Dirichlet unit theorem (see e.g. [Mil09]) one can prove that $l(O_k^{*,+})$ is a full lattice in L . Therefore there exists a group U that is admissible.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \text{im } z > 0\}$. Let $U \subset O_k^{*,+}$ be a group which is admissible for K . Let O_K be an additive group of algebraic integers. The group U acts on O_K multiplicatively. This defines a structure of the semidirect product $U' := U \ltimes O_K$. Define the action of U' on $\mathbb{H}^s \times \mathbb{C}^t$ as follows. The element $u \in U$ acts on $\mathbb{H}^s \times \mathbb{C}^t$ mapping (z_1, \dots, z_m) to $(\sigma_1(u)z_1, \dots, \sigma_m(u)z_m)$. Since U lies in $O_k^{*,+}$, the action U on the first s coordinates preserves \mathbb{H} .

The additive group O_K acts on $\mathbb{H}^s \times \mathbb{C}^t$ by parallel translations: $a \in O_K$ is mapping (z_1, \dots, z_m) to $(\sigma_1(a) + z_1, \dots, \sigma_m(a) + z_m)$. Since the first s embeddings are real, this action preserves \mathbb{H} in the first s coordinates.

One can see that $(u, a) \in U \ltimes O_K$ maps (z_1, \dots, z_m) to $(\sigma_1(u)z_1 + \sigma_1(a), \dots, \sigma_m(u)z_m + \sigma_m(a))$. Let us show that this action is compatible with the group operation in the semidirect product.

By definition, one has $(u, a) * (u_1, a_1) = (uu_1, ua_1 + a)$, and

$$\begin{aligned} (u, a) \circ (u_1, a_1)(z_1, \dots, z_m) &= (u, a)(\sigma_1(u_1)z_1 + \sigma_1(a_1), \dots, \sigma_m(u_1)z_m + \sigma_m(a_1)) = \\ &= (\sigma_1(u)\sigma_1(u_1)z_1 + \sigma_1(u)\sigma_1(a_1) + \sigma_1(a), \dots, \sigma_m(u)\sigma_m(u_1)z_m + \sigma_m(u)\sigma_m(a_1) + \sigma_m(a)) = \\ &= (\sigma_1(uu_1)z_1 + \sigma_1(ua_1 + a), \dots, \sigma_m(uu_1)z_m + \sigma_m(ua_1 + a)) = (uu_1, ua_1 + a)(z_1, \dots, z_m) = \\ &= ((u, a) * (u_1, a_1))(z_1, \dots, z_m), \end{aligned}$$

This proves the compatibility.

Definition 1.4: *Oeljeklaus-Toma manifold* is a quotient of $\mathbb{H}^s \times \mathbb{C}^t$ by the action of the group $U \ltimes O_K$ which was defined above.

This quotient exists because $U \ltimes O_K$ acts properly discontinuously on $\mathbb{H}^s \times \mathbb{C}^t$. Therefore $\mathbb{H}^s \times \mathbb{C}^t / U \ltimes O_K$ is a compact complex manifold. To prove it, let U be admissible for K . The quotient $\mathbb{H}^s \times \mathbb{C}^t / \sigma(O_K)$ is obviously diffeomorphic to the trivial toric bundle $(\mathbb{R}_{>0})^s \times (S^1)^n$. The group U acts properly discontinuously on the base $(\mathbb{R}_{>0})^s$. Therefore it acts properly discontinuously on $\mathbb{H}^s \times \mathbb{C}^t / \sigma(O_K)$. Also, groups U and O_K act holomorphically on $\mathbb{H}^s \times \mathbb{C}^t$. Therefore the quotient has a holomorphic structure.

2 Curves on the Oeljeklaus-Toma manifolds

In this section we shall prove that there are no complex curves on the Oeljeklaus-Toma manifolds, just as on the Inoue surfaces (see [I]).

2.1 The exact positive (1,1)-form on the Oeljeklaus-Toma manifold

Let M be a smooth complex manifold, z_1, \dots, z_n — local complex coordinates in the open neighborhood of the point $x \in M$.

Definition 2.1: The *Hodge decomposition* of the Grassmanian algebra $\Lambda^* M$ is a decomposition into the direct sum $\Lambda^* M = \oplus_{p,q} \Lambda^{p,q} M$, where $\Lambda^{p,q} M = \Lambda^{p,0} M \wedge \Lambda^{0,q} M$; space $\Lambda^{p,0} M$ of the real differential forms is generated by $dz_{i_1} \wedge \dots \wedge dz_{i_p}$, and $\Lambda^{0,q} M$ is generated by $d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_q}$.

Therefore $\Lambda^2 M = \Lambda^{2,0} M \oplus \Lambda^{1,1} M \oplus \Lambda^{0,2} M$.

Definition 2.2: $(1,1)$ -form on a complex manifold M is a section of $\Lambda^{1,1} M$.

Definition 2.3: $(1,1)$ -form ω on a complex manifold M is *positive (or semipositive)* if $\sqrt{-1}\omega(x, \bar{x}) \geq 0$ for each tangent vector $x \in T_m^{1,0} M$.

As in [O–V], we consider a certain positive $(1,1)$ -form on Oeljeklaus-Toma manifold $M = \mathbb{H}^s \times \mathbb{C}^t / (U \ltimes O_K)$. Firstly, we introduce a $(1,1)$ -form $\tilde{\omega}$ on $\widetilde{M} = \mathbb{H}^s \times \mathbb{C}^t$ which is preserved by the action of the group $\Gamma = (U \ltimes O_K)$ and since then it would be a $(1,1)$ -form on M .

Let (z_1, \dots, z_m) be complex coordinates on \widetilde{M} . Define $\varphi(z) = \prod_{i=1}^s \text{im}(z_i)$. Since the first s components of \widetilde{M} correspond to upper half-planes $\mathbb{H} \subset \mathbb{C}$, this function is positive on \widetilde{M} .

Let us now consider a form $\tilde{\omega} = \sqrt{-1} \partial \bar{\partial} \log \varphi$. Using standard coordinates on \widetilde{M} one can write this form as $\tilde{\omega} = \sqrt{-1} \sum_{i=1}^s \frac{dz_i \wedge d\bar{z}_i}{(\text{im } z_i)^2}$. Therefore $\tilde{\omega}$ is a positive $(1,1)$ -form on \widetilde{M} .

Let us show that this form is Γ -invariant.

We denote by $\Gamma = (U \ltimes O_K)$. Γ is a semidirect product of the additive group O_K and the multiplicative group U . Additive group acts on the first s components of \widetilde{M} (which correspond to upper half-planes $\mathbb{H} \subset \mathbb{C}$) by translations along the real line. Therefore it does not change $\text{im } z_i$ for $i = 1 \dots s$. Hence the function $\log \varphi$ is preserved by the action of the additive component.

The multiplicative component acts on the first s coordinates \widetilde{M} by multiplying them by a real number (since the first s embeddings of the number field K are real). Then every $\text{im } z_i$ is multiplied by a real number and so there is a real number added to $\log(\text{im } z_i)$. Since $\log \varphi(z) = \sum_{i=1}^s \log(\text{im } z_i)$, there is a real number added to $\log \varphi$. Operator $\bar{\partial}$ is zero on the constants, so $\tilde{\omega} = \sqrt{-1} \partial \bar{\partial} \log \varphi$ is preserved by action of group Γ .

Since $(1, 1)$ -form $\tilde{\omega}$ is Γ -invariant it is a pullback of $(1, 1)$ -form ω on the Oeljeklaus-Toma manifold $M = \widetilde{M}/\Gamma$.

Let's now show that the form $\tilde{\omega}$ is exact on \widetilde{M} . For that we define operator d^c .

Definition 2.4: Define the *twisted differential* $d^c = I^{-1}dI$ where d is a De Rham differential and I is the operator of the almost complex structure.

Since $dd^c = 2\sqrt{-1}\partial\bar{\partial}$ (see [G-H]), one can see that $\tilde{\omega} = \sqrt{-1}\partial\bar{\partial} \log \varphi = \frac{1}{2}dd^c \log \varphi$ and so $\tilde{\omega}$ is exact as a form on \widetilde{M} . Also since the operator d^c vanishes on the constants the form $d^c \log \varphi$ is Γ -invariant, so ω is exact on M .

2.2 $(1, 1)$ -form ω and curves on the Oeljeklaus-Toma manifold

Since the form ω on manifold M is positive, its integral on any complex curve $C \subset M$ is nonnegative. The form ω is exact. Hence the Stokes' theorem implies that its integral on any complex curve vanishes. So, if $C \subset M$ is a closed complex curve, ω vanishes on it.

To find out on which curves ω vanishes, let us define the zero foliation of the form ω .

Definition 2.5: *Involutive distribution (or foliation)* on M is a subbundle $B \subset TM$ in the tangent bundle that is closed under commutator: $[B, B] \subset B$.

Definition 2.6: A *leaf of a foliation* is a submanifold in M such that its dimension equal $\dim B$ and that is tangent to B at every point (not necessarily closed).

Theorem 2.7: (Frobenius) Let $B \subset TM$ be an involutive distribution. Then for each point of a manifold M , there is at most one leaf of this distribution that contains this point (see e.g. [Boo] Section IV. 8. Frobenius's Theorem).

Definition 2.8: The *zero foliation* of $(1, 1)$ -form ω on M is a subbundle of TM that consists of tangent vectors $x \in TM$ such that $\omega(x, Ix) = 0$ where I is an operator of complex structure.

Consider the zero foliation of $\tilde{\omega}$ on \widetilde{M} .

The form $\tilde{\omega}$ is strictly positive on each vector $v = (z_1, \dots, z_n)$ such that at least one of z_i is nonzero. Such a vector can not be in the leaf of the zero foliation. Therefore on each leaf of the zero foliation of the form $\tilde{\omega}$ the first s coordinates are constant.

Hence a leaf of the zero foliation of $\tilde{\omega}$ on \widetilde{M} is isomorphic to \mathbb{C}^t .

Let us now consider the zero foliation of ω on M .

We show that the image of the action of Γ on any leaf L of the zero foliation of the form $\tilde{\omega}$ does not intersect with L .

One can see that L is $(z_1, \dots, z_s) \times \mathbb{C}^t$ for some fixed (z_1, \dots, z_m) . Therefore, for any $\gamma \in \Gamma$ such that $L \cap \gamma(L) \neq \emptyset$, the first s coordinates of the points in L coincide with the first s coordinates of the points in $\gamma(L)$. Therefore we have a following system of equations:

$$\sigma_i(u)z_i + \sigma_i(a) = z_i, \quad i = 1 \dots s,$$

where $\gamma = (u, a)$.

These equations imply that $z_i = \frac{\sigma_i(a)}{1 - \sigma_i(u)}$. Therefore z_i is real but \mathbb{H} doesn't have real elements.

Therefore $L \cap \Gamma(L) = \emptyset$.

Since ω vanishes on each compact curve $C \subset M$, each curve is contained in some leaf of the zero foliation of ω . Since ω is Γ -invariant, each leaf of the zero foliation of ω on M is a factor of the leaf of the zero foliation of $\tilde{\omega}$ on \widetilde{M} . Therefore, it is isomorphic to \mathbb{C}^t . All the coordinate functions

$z_i, i = 1, \dots, t$ are holomorphic and therefore constant on all compact connected subvarieties. Therefore, \mathbb{C}^t does not contain complex curves, and M does not contain complex curves either because it's compact and every closed curve on a compact manifold is compact.

We proved the following theorem:

Theorem 2.9: There are no closed complex curves on the Oeljeklaus-Toma manifolds.

References

- [Aus] Auslander L. *The structure of compact locally affine manifolds*. Topology 3 (1964), 131-139.
- [B1] Bieberbach L. *Über die Bewegungsgruppen der Euklidischen Räume I*. Mathematische Annalen 70 (3), 297-336.
- [B2] Bieberbach L. *Über die Bewegungsgruppen der Euklidischen Räume II*. Mathematische Annalen 72 (3), 400-412.
- [Boo] Boothby W.M. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Academic Press, San Diego, California, 2003.
- [G-H] Griffiths Ph., Yarris J. *Principles of Algebraic Geometry*. Wiley-Interscience, 1994.
- [I] Inoue M. *On surfaces of Class VII₀*, Invent. Math. 24 (1974), 269-310.
- [Mil08] Milne J.S. *Fields and Galois Theory, September 2008*.
This paper can be found on <http://www.jmilne.org/math/CourseNotes/ft.html>, version 4.21
- [Mil09] Milne J.S. *Algebraic Number Theory, April 2009*.
This paper can be found on <http://www.jmilne.org/math/CourseNotes/ant.html>, version 3.02
- [O-T] Oeljeklaus K., Toma M. *Non-Kähler compact complex manifolds associated to number fields*. Ann. Inst. Fourier 55 (2005), 1291-1300.
- [O-V] Ornea L., Verbitsky M. *Subvarieties in Oeljeklaus-Toma manifolds*.
- [P-V] Parton M., Vuletescu V. *Examples of non-trivial rank in locally conformal Kähler geometry*. Math. Z. (2010), DOI 10.1007/s00209-010-0791-5, arXiv:1001.4891.
- [R] Raghunathan M.S. *Discrete subgroups of Lie groups*. Springer 1972.
- [V] Voisin C. *Hodge Theory and Complex Algebraic Geometry Volume 1*. Cambridge University Press, 2002.

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